Summation techniques

Convolution method: A trick to compute partial sums of a Dirichlet convolution.

Main idea: (change order of summation)

Suppose $g, h \in \mathcal{A}$. Then

$$\sum_{n\le x} g*h(n) = \sum_{n\le x}\sum_{d|n} g(d)h\left(\frac{n}{d}\right)$$

$$= \sum_{d\le x} g(d) \sum_{\substack{n\equiv 0(d)\\ n\le x}} h\left(\frac{n}{d}\right)$$

$d|n \Rightarrow d\le x$

$$= \sum_{d\le x} g(d) \sum_{m\le \frac{x}{d}} h(m).$$

Example: $\displaystyle\sum_{n\le x}\varphi(n) = Cx^2 + O(x\log x)$

We know $\varphi = \mu * \mathrm{id}$.

$$\Rightarrow \sum_{n\le x}\varphi(n) = \sum_{d\le x}\mu(d)\sum_{m\le\frac{x}{d}} m$$

But $\displaystyle\sum_{m\le\frac{x}{d}} m = \frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)$ (exercise).

$$\Rightarrow \sum_{n\le x}\varphi(n) = \sum_{d\le x}\mu(d)\cdot\left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + O\left(\frac{x}{d}\right)\right).$$

$$= \frac{x^2}{2} \sum_{d \le x} \frac{\mu(d)}{d^2} + O\Big(\sum_{d \le x} \mu(d) \frac{x}{d} \Big). \quad (*)$$

Note that $\displaystyle\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}$ convergent, and

$$\sum_{d \le x} \frac{\mu(d)}{d^2} = \underbrace{\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}}_{=:\,C} - \sum_{d > x} \frac{\mu(d)}{d^2}$$

$$= C + O\Big(\sum_{d > x} \frac{\mu(d)}{d^2} \Big) = C + O\Big(\sum_{d > x} \frac{1}{d^2} \Big)$$

$$= C + O\Big(\frac{1}{x} \Big). \quad (**)$$

$\le \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x}$.

Also $\displaystyle\Big| \sum_{d \le x} \mu(d) \frac{x}{d} \Big| \le x \sum_{d \le x} \frac{1}{d} \ll x \log x.$ (***)

From (*), (**), (***), we get

$$\sum_{n \le x} \varphi(n) = \frac{x^2}{2} \Big(C + O\Big(\frac{1}{x}\Big) \Big) + O(x \log x)$$

$$= x^2 \cdot \frac{C}{2} + O(x \log x).$$

Summation formulae

Proposition (Approximating sum by integral)
Let $f:[y,x]\to\mathbb{R}$ monotonic, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + O(|f(x)|+|f(y)|)$$

Examples:
(i)
$$\sum_{1\leq n\leq x}\frac{1}{n} = 1+\sum_{1<n\leq x}\frac{1}{n} =$$
$$= 1+\int_1^x \frac{1}{t}\,dt + O\left(1+\frac{1}{x}\right)$$
$$= \log x + O(1).$$

(ii)
$$\sum_{1\leq n\leq x}\log n = x\log x - x + O(\log x).$$

(iii)
$$\sum_{n\leq x} n = \frac{x^2}{2}+O(x).$$

Proof of proposition:
Case 1: $y,x\in\mathbb{Z}$, f increasing.

We use that
$$\int_{n-1}^{n} f(t)\,dt \leq f(n) \leq \int_{n}^{n+1} f(t)\,dt$$

Hence $\displaystyle\int_y^x f(t)\,dt \le \sum_{y<n\le x} f(n) \le f(x) + \int_{y+1}^x f(t)\,dt.$

$$\Rightarrow 0 \le \sum_{y<n\le x} f(n) - \int_y^x f(t)\,dt$$

$$= -\int_y^{y+1} f(t)\,dt + \Big(\sum_{y<n\le x} f(n) - \int_{y+1}^x f(t)\,dt\Big)$$

$$\le \left|\int_y^{y+1} f(t)\,dt\right| + f(x)$$

$$\le 2(|f(x)| + |f(y)|).$$

{since $|f(t)| \le \max\{|f(x)|, |f(y)|\}$ for $t \in [y,x]$}

Case 2: y, x general, f increasing.

From previous case,

$$\sum_{y<n\le x} f(n) = \int_{\lfloor y\rfloor+1}^{\lfloor x\rfloor} f(t)\,dt + O(|f(\lfloor y\rfloor+1)| + |f(\lfloor x\rfloor)|)$$

$$= \int_y^x f(t)\,dt + O\left(\left|\int_y^{\lfloor y\rfloor+1} f(t)\,dt\right| + \left|\int_{\lfloor x\rfloor}^x f(t)\,dt\right|\right)$$

$$+ O(|f(y)| + |f(x)|)$$

$$= \int_y^x f(t)\,dt + O\left(|f(y)| + |f(x)|\right) \checkmark$$

Case 3: y, x general, f decreasing

Let $F = -f$, so F increasing, so

$$\sum_{y<n\leq x} f(n) = -\sum_{y<n\leq x} F(n) = -\int_y^x F(t)\,dt + O(|F(x)| + |F(y)|)$$

$$= \int_y^x f(t)\,dt + O\left(|f(x)| + |f(y)|\right).$$

$\square$

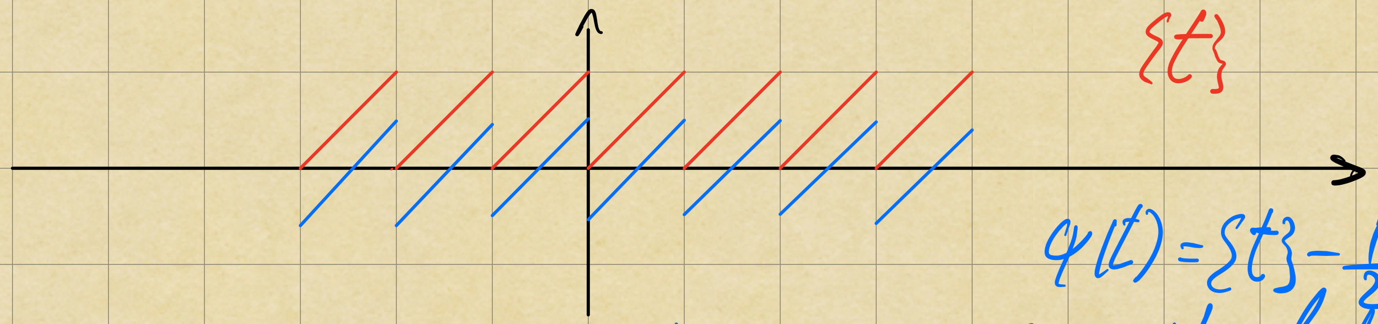
Sometimes we have better approximations for general class of functions:

Theorem (Euler - Maclaurin formula).
Let $f : [y, x] \to \mathbb{C}$ a C^1-function. Then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x f'(t)\psi(t)\,dt + f(y)\psi(y) - f(x)\psi(x),$$

where $\psi(t) = \{t\} - \frac{1}{2}$.

Exact formula!



$\psi(t) = \{t\} - \frac{1}{2}$.

has mean 0 over every interval of length 1.

Proof:

Claim: Let $n \in \mathbb{Z}$. Then

$$\frac{f(n)+f(n+1)}{2} = \int_n^{n+1} f(t)\,dt + \int_n^{n+1} \psi(t) f'(t)\,dt.$$

Proof of claim: $\displaystyle\int_n^{n+1} \psi(t) f'(t)\,dt = \int_n^{n+1} \left(t - n - \tfrac{1}{2}\right) f'(t)\,dt$

$$= \int_n^{n+1} t f'(t)\,dt - \left(n+\tfrac{1}{2}\right) \int_n^{n+1} f'(t)\,dt$$

$$= \Big[t f(t)\Big]_n^{n+1} - \int_n^{n+1} f(t)\,dt - \left(n+\tfrac{1}{2}\right) \Big[f(t)\Big]_n^{n+1}$$

$$= (n+1) f(n+1) - n f(n) - \int_n^{n+1} f(t)\,dt - \left(n+\tfrac{1}{2}\right)\big(f(n+1) - f(n)\big)$$

$$= \frac{f(n+1)}{2} + \frac{f(n)}{2} - \int_n^{n+1} f(t)\,dt. \quad \checkmark$$

Case 1: y, x integers. We sum the quantities in the claim for all integers $y \le n < x$.

We get $\frac{f(y)}{2} + \sum_{y<n<x} f(n) + \frac{f(x)}{2} = \int_y^x f(t)dt + \int_y^x \psi(t) f'(t)$

$$\Rightarrow \sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x \psi(t) f'(t) + \frac{f(x)}{2} - \frac{f(y)}{2}$$

claim follows from the fact that $\psi(k) = -\frac{1}{2}$, for $k \in \mathbb{Z}$.

Case 2 : General y, x.

Exercise : $\int_y^{\lfloor y\rfloor+1} \psi(t) f'(t)dt = -\psi(y) f(y) + \frac{f(\lfloor y\rfloor+1)}{2} - \int_y^{\lfloor y\rfloor+1} f(t)dt$

$$\int_{\lfloor x\rfloor}^x \psi(t) f'(t) = \psi(x) f(x) + \frac{f(\lfloor x\rfloor)}{2} - \int_{\lfloor x\rfloor}^x f(t)dt.$$

$\boxtimes$

Application : $\sum_{1\leq n\leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x})$.

$\hookrightarrow$ Euler constant

Recall: Monotonicity only gave us $\log(x) + O(1)$.

By Euler-Maclaurin: $\sum_{1\leq n\leq x} \frac{1}{n} = 1 + \sum_{1<n\leq x} \frac{1}{n}$

$$= 1 + \int_1^x \left(\frac{1}{t} - \frac{\psi(t)}{t^2}\right)dt - \frac{1}{2} - \frac{\psi(x)}{x}$$

$$= \log x + \underbrace{\left(\frac{1}{2} - \int_1^\infty \frac{\psi(t)}{t^2}\,dt\right)}_{\text{This is the constant } \gamma.} + \int_x^\infty \frac{\psi(t)}{t^2}\,dt - \underbrace{\frac{\psi(x)}{x}}_{O(\frac{1}{x})}$$

Now $\left| \int_x^\infty \frac{\psi(t)}{t^2}\,dt \right| \ll \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x} = O\left(\frac{1}{x}\right)$.

Theorem (Abel summation or Partial summation)

Let $f \in \mathbb{R}$, $0 < y < x$, $g \in C^1([y,x])$.

Define $F(T) := \sum_{n \leq T} f(n)$.

Then $\sum_{y < n \leq x} f(n) g(n) = F(x)g(x) - F(y)g(y) - \int_y^x F(t)g'(t)\,dt$

Proof: WLOG $\lfloor y \rfloor + 1 \leq \lfloor x \rfloor$

(otherwise no integer between y and x, LHS $= 0$ and $F(x) = F(y) = F(t)$, $\forall t \in [y, x]$, so RHS $= 0$).

Then $\sum_{y < n \leq x} f(n) g(n) = \sum_{\lfloor y \rfloor + 1 \leq n \leq \lfloor x \rfloor} (F(n) - F(n-1)) g(n)$

$$= \sum_{\lfloor y \rfloor + 1 \leq n \leq \lfloor x \rfloor} F(n) g(n) - \sum_{\lfloor y \rfloor \leq n \leq \lfloor x \rfloor - 1} F(n) g(n+1)$$

$$= F(x)\,g(\lfloor x\rfloor) - F(y)\,g(\lfloor y\rfloor+1)$$

$$- \sum_{\lfloor y\rfloor+1 \le n \le \lfloor x\rfloor-1} \underbrace{F(n)\big(g(n+1)-g(n)\big)}_{\int_n^{n+1} F(\xi)\,g'(\xi)\,d\xi}.$$

$$= F(x)\,g(\lfloor x\rfloor) - F(y)\,g(\lfloor y\rfloor+1) - \int_{\lfloor y\rfloor+1}^{\lfloor x\rfloor} F(t)\,g'(t)\,dt$$

Now we see $\displaystyle\int_{\lfloor x\rfloor}^{x} F(t)\,g'(t)\,dt = F(x)\big(g(x)-g(\lfloor x\rfloor)\big)$.

$$\int_{y}^{\lfloor y\rfloor+1} F(t)\,g'(t)\,dt = F(y)\big(g(\lfloor y\rfloor+1)-g(y)\big).$$

$\blacksquare$

<u>Corollary</u>: Let $g : [y,\infty) \to \mathbb{C}$, $f \in \mathcal{R}$ and suppose $F(x)\,g(x) \to 0$ as $x \to \infty$.

Then
$$\sum_{n \ge y} f(n)\,g(n) = -\int_{y}^{\infty} F(t)\,g'(t)\,dt - F(y)\,g(y)$$
whenever both sides converge.

Example: $\exists\, C' > 0$ s.t. $\displaystyle\sum_{n \le x} \frac{\varphi(n)}{n} = C'x + O((\log x)^2)$.

Pf: Denote $F(y) = \displaystyle\sum_{n \le y} \varphi(n)$. By earlier today,

$$F(y) = C \cdot y^2 + O(y \log y).$$

Hence by partial summation,

$$\sum_{n \le x} \frac{\varphi(n)}{n} = 1 + \frac{F(x)}{x} - 1 + \int_1^x F(t) \cdot \frac{1}{t^2}\, dt$$

$$= \frac{Cx^2 + O(x \log x)}{x} + \int_1^x \frac{c \cdot t^2 + O(t \log t)}{t^2}\, dt$$

$$= C \cdot x + O(\log x) + C(x-1) + O\left(\int_1^x \frac{\log t}{t}\, dt\right)$$

$$= 2 \cdot C \cdot x + O(\log x) + O\left(\left[\tfrac{1}{2}(\log t)^2\right]_1^x\right)$$

$$= 2 \cdot C \cdot x + O((\log x)^2).$$